Pergamon

PII: S0021-8928(98)00062-8

J. Appl. Maths Mechs, Vol. 62, No. 3, pp. 483–488, 1998

0021-8928/98/$—see front matter

UNSTEADY VISCOUS FLOW IN THREE DIMENSIONS AND AROUND PLANE SURFACES†

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(*Received* 2 *April* 1997)

Exact solutions are obtained for certain problems of the unsteady flow of an incompressible viscous fluid in the case of velocity distributions for which the Navier–Stokes equations become linear. Unlike previous solutions of these problems, more general boundary conditions and flow non-uniformity are taken into account. The smoothing of a velocity discontinuity and vorticity propagation in three dimensions is considered as well as fluid flow over an infinite plate, in an infinite two-sided corner and between parallel plates. © 1998 Elsevier Science Ltd. All rights reserved.

1. SMOOTHING OF A VELOCITY DISCONTINUITY AND VORTICITY PROPAGATION

In the rectangular Cartesian system of coordinates $Ox_1x_2x_3$, the velocity components in the three directions are taken to be of the form

$$u_1 = u_1(x_3,t),\ u_2 = u_2(x_3,t),\ u_{30} = \text{const}$$

The incompressibility condition is then satisfied identically, and the Navier–Stokes equations take the form

$$\frac{\partial u_k}{\partial t} + u_{30}\frac{\partial u_k}{\partial x_3} = -\frac{1}{\rho}\frac{\partial p}{\partial x_k} + \nu\frac{\partial^2 u_k}{\partial x_3^2},\ k = 1,2; \rho, \nu = \text{const} \tag{1.1}$$

For the motion of a fluid which occupies the whole space, the initial velocity distributions are assumed known

$$u_k(x_3,0) = \varphi_k(x_3),\ k = 1,2 \tag{1.2}$$

The pressure in the whole space is taken to be constant ($\delta p/\delta x_k \equiv 0$, $k = 1, 2$).

Together with the fixed system of coordinates $Ox_1x_2x_3$, we introduce a moving system $O^\circ x_1^\circ x_2^\circ x_3^\circ$ for which $x_1^\circ = x_1$, $x_2^\circ = x_2$, $x_3^\circ = x_3 - u_{30}t$, $t^\circ = t$. In the new variables, the equations contain no terms of the form of the second terms of the left-hand side of (1.1). Solving Cauchy's problem [1] for these equations, which are autonomous, with initial conditions $u_k(x_3, 0) = \varphi_k(x_3^\circ)$ and reverting to the original variables, we obtain the solution of problem (1.1), (1.2)

$$u_k(x_3,t) = \int_{-\infty}^{\infty} G(x_3 - u_{30}t, \xi, t)\varphi_k(\xi)d\xi,\ k = 1,2$$

$$G(x_3 - u_{30}t, \xi, t) = \frac{1}{2\sqrt{\pi\nu t}}\exp\left[-\frac{(x_3 - u_{30}t - \xi)^2}{4\nu t}\right] \tag{1.3}$$

For $t > 0$ Poisson's integral (1.3) represents a bounded solution of the equation for any bounded function $|\varphi(\xi)| < M$ which, when $t = 0$, continuously adjoins $\varphi_k(x_3)$ at all points of continuity of that function; the function $\varphi_k(x_3)$ can have a finite number of points of discontinuity of the first kind.

Suppose the initial velocities have constant but different values for $x_3 > 0$ and $x_3 < 0$

$$u_k(x_3,0) = \varphi_k(x_3) = \begin{cases} U_k, & x_3 > 0 \\ V_k, & x_3 < 0;\ k = 1,2 \end{cases}$$

In that case formulae (1.3) become [1]

†*Prikl. Mat. Mekh.* Vol. 62, No. 3, pp. 519–525, 1998.

$$u_k(x_3,t)=[U_k+V_k+(U_k-V_k)\Phi(z)]/2,\ k=1,2$$
$$\Phi(z)=\frac{2}{\sqrt{\pi}}\int_0^z \exp(-\alpha^2)d\alpha,\ z=\frac{x_3-u_{30}t}{2\sqrt{\nu t}} \tag{1.4}$$

For the solutions obtained in [2, 3] the plane $x_3 = 0$ was a surface of discontinuity of the velocity in one direction ($u_2 \equiv 0$). In contrast, formulae (1.4) give a solution for the smoothing of a velocity discontinuity in a flow with known constant velocity $u_3 = u_{30}$ in which, initially, the angle between the two uniform flows is, generally speaking, different from 0 and π, that is, the problem is non-planar.

We shall now assume that

$$u_2=u_{20}=\text{const},\ u_3=u_{30}=\text{const},\ u_1=u_1(x_2,x_3,t)$$

Then the incompressibility condition is satisfied identically, and the Navier–Stokes equations give

$$\frac{\partial u_1}{\partial t}+u_{20}\frac{\partial u_1}{\partial x_2}+u_{30}\frac{\partial u_1}{\partial x_3}=-\frac{1}{\rho}\frac{\partial p}{\partial x_1}+\nu\left(\frac{\partial^2 u_1}{\partial x_2^2}+\frac{\partial^2 u_1}{\partial x_3^2}\right) \tag{1.5}$$

For a fluid which occupies the whole space, we shall assume that the pressure is constant over the whole space and the initial velocity distribution is known

$$u_1(x_2,x_3,0)=\zeta(x_2,x_3) \tag{1.6}$$

By analogy with (1.3) we find the solution of problem (1.5), (.6)

$$u_1(x_2,x_3,t)=\int_{-\infty}^{\infty}\int G(x_2-u_{20}t,x_3-u_{30}t,t,\xi_2,\xi_3)\zeta(\xi_2,\xi_3)d\xi_2 d\xi_3 \tag{1.7}$$

$$G(y_2,y_3,t,\xi_2,\xi_3)=\frac{1}{4\pi\nu t}\exp\left[-\frac{(y_2-\xi_2)^2+(y_3-\xi_3)^2}{4\nu t}\right]$$

Let

$$\zeta(x_2,x_3)=\begin{cases}U, & (x_2,x_3)\in Q=\{0\leqslant x_2<\infty,\ 0\leqslant x_3<\infty\}\\ 0, & \forall(x_2,x_3)\notin Q\end{cases}$$

After some reduction, instead of (1.7) we obtain

$$u_1(x_2,x_3,t)=\frac{U}{4}\left[1+\Phi\left(\frac{x_2-u_{20}t}{2\sqrt{\nu t}}\right)\right]\left[1+\Phi\left(\frac{x_3-u_{30}t}{2\sqrt{\nu t}}\right)\right]$$

This is a solution of the smoothing problem in a flow with velocity components $u_2 = u_{20}$, $u_3 = u_{30}$ where a uniform flow which occupies an infinite two-sided corner is in contact with a fluid at rest.

For the vorticity propagation problem, we shall assume that the projections of the velocity onto the axes of cylindrical coordinates r θ, z at the initial time have the following values

$$u_r=a_1\cos\theta+a_2\sin\theta,\ u_\theta=\alpha(r)-a_1\sin\theta+a_2\cos\theta,\ u_z=\beta(r)+a_3$$

where a_1, a_2, a_3 = const.

In this case the vorticity distribution at the initial time is given by the formulae

$$\operatorname{rot}u=\{\Omega_r,\Omega_\theta,\Omega_z\},\ \Omega_r=0,\ \Omega_\theta=-\frac{d\beta}{dr},\ \Omega_z=\frac{1}{r}\frac{d(r\alpha(r))}{dr}\equiv\omega(r)$$

On the basis of the form of the initial conditions, we shall assume that $u_k = u_k(x_1, x_2, t)$ ($k = 1, 2, 3$), and represent the velocities in the form $u_k = u_k(x_1, x_2, t)\ v_k(x_1, x_2, t) + a_k (k = 1, 2, 3)$. We introduce a moving system of coordinates $O^\circ x_1^\circ x_2^\circ x_3^\circ$ for which $x_1^\circ = x_1 - a_1 t$, $x_2^\circ = x_2 - a_2 t$, $x_3^\circ = x_3$, $t^\circ = t$. Then, changing to cylindrical coordinates r°, θ°, z° in this moving system of coordinates, we find that the equations for $v_{z^\circ} = v_{z^\circ}(r^\circ, t^\circ)$ and $\Omega_{z^\circ} = (z^\circ)^{-1}\ \partial(r^\circ v_{\theta^\circ}(r^\circ, t^\circ))/\partial r^\circ$ are autonomous and of the same form. Moreover, the equation for z° is similar to the equation for the analogous vorticity component in the case of the vorticity diffusion in a medium at rest. Thus, using the known solution [2], we have

$$\Omega_{z^\circ}(r^\circ,t^\circ)=\frac{1}{2\nu t^\circ}\exp\left(-\frac{r^{\circ 2}}{4\nu t^\circ}\right)\int_0^\infty \omega(s)\exp\left(-\frac{s^2}{4\nu t^\circ}\right)J_0\left(\frac{sr^\circ}{2\nu t^\circ}\right)sds$$

and a similar expression for $v_{z^\circ}(r^\circ, t^\circ)$ (with $\omega(s)$ replaced by $\beta(s)$), where J_0 is a zero-order Bessel function of the first kind.

Writing the solution as the projection of the velocity onto the axes of cylindrical coordinates, we obtain

$$u_r(r,\theta,t)=v_{\theta^\circ}(r^\circ,t^\circ)\sin(\theta-\theta^\circ)+a_1\cos\theta+a_2\sin\theta \tag{1.8}$$

$$u_\theta(r,\theta,t)=v_{\theta^\circ}(r^\circ,t^\circ)\cos(\theta-\theta^\circ)-a_1\sin\theta+a_2\cos\theta$$

$$u_z(r,\theta,t)=v_{z^\circ}(r^\circ,t^\circ)+a_3$$

$$r^\circ=\left[r^2-2rt^\circ(a_1\cos\theta+a_2\sin\theta)+\left(a_1^2+a_2^2\right)t^{\circ 2}\right]^{1/2},\quad \mathrm{tg}\theta^\circ=\frac{r\sin\theta-a_2t^\circ}{r\cos\theta-a_1t^\circ}$$

$$v_{\theta^\circ}(r^\circ,t^\circ)=\frac{1}{r^\circ}\int_0^{r^\circ}\Omega_{z^\circ}(r^\circ,t^\circ)r^\circ dr^\circ,\quad t^\circ=t$$

The vorticity distribution is found from the formulae

$$\Omega_r\equiv 0,\ \Omega_\theta=-\frac{\partial u_z}{\partial r},\ \Omega_z=\frac{1}{r}\left[\frac{\partial(ru_\theta)}{\partial r}-\frac{\partial u_r}{\partial\theta}\right] \tag{1.9}$$

Thus, whereas the solution obtained in [2] was for the plane problem of vorticity diffusion in a medium at rest ($a_1, a_2, a_3\equiv 0$, $\beta(r)\equiv 0$, formulae (1.8) and (1.9) give the solution of the problem of vorticity propagation in drifting flow.

2. THE FLUID MOTION OF A LIQUID OVER A PLATE

We will now consider the flow over an infinite plane surface $x_3 = 0$: it is required to find a solution of Eq. (1.1) ($u_{30} = 0$) in the region $0 < x_3 < \infty$, $0 < t$ which satisfies the conditions

$$u_k(x_3,0)=\varphi_k(x_3),\ k=1,2,\ 0\leqslant x_3<\infty$$

$$u_k(0,t)=\mu_k(t),\ u_k(\infty,t)=U_k(t),\ k=1,2,\ 0\leqslant t$$

This is a more general problem than the plane problem [3, 4], in which the plane boundary is suddenly put into motion in a fluid at rest, or a plane wall performs rectilinear harmonic oscillations in its own plane in a fluid at rest. In this case allowance is made for arbitrary initial conditions, the motion of the wall and the "free-stream velocity" far from the wall, when the angle between the "free-stream velocity" and the wall velocity is possibly different from 0 or π, so that the problem is non-planar.

The solution can be represented in the form

$$u_k(x_3,t)=v_k(x_3,t)+U_k(t)-U_k(0),\ v_k(x_3,t)=v_{k1}(x_3,t)+v_{k2}(x_3,t)$$

$$v_{k1}(x_3,t)=\int_0^\infty g(x_3,\xi,t)\varphi_k(\xi)d\xi,\ v_{k2}(x_3,t)=2\nu\int_0^t\frac{\partial G(x_3,0,t-\tau)}{\partial\xi}\alpha_k(\tau)d\tau$$

$$g(x_3,\xi,t)=\frac{1}{2\sqrt{\pi\nu t}}\left\{\exp\left[-\frac{(x_3-\xi)^2}{4\nu t}\right]-\exp\left[-\frac{(x_3+\xi)^2}{4\nu t}\right]\right\}$$

$$\alpha_k(t)=\mu_k(t)+U_k(0)-U_k(t),\ -\frac{dU_k}{dt}=\frac{1}{\rho}\frac{\partial p}{\partial x_k},\ k=1,2$$

Suppose that the plate is not only moving in its own plane at a velocity $\{\mu_1(t), \mu_2(t)\}$, but also in a perpendicular direction with constant velocity u_{30}. The boundary conditions will then be

$$u_k(u_{30}t,t)=\mu_k(t),\ k=1,2,0\le t$$

In a moving system of coordinates, the problem reduces to the previous one. Therefore, one solution is given by the above relations, with $x_3 - u_{30}t$ instead of x_3.

The functions $\upsilon_{k2}(x_3, t)$, which allow for the motion of the wall, are defined for any bounded piecewise-continuous functions $\alpha(t)$ [1]. It is thus possible to model the various flows caused, in particular, by the wall that starts to move or stops abruptly. Suppose, for example, that the plane wall, previously at rest, suddenly starts to move in its own plane at a constant velocity μ_1 in the direction of the Ox_1 axis. At a time $t = t_1$, the wall suddenly stops, and at a time $t = t_2$, $t_2 > t_1$, it suddenly starts to move in its own plane at a constant velocity μ_2 in the Ox_2 direction.

In this case, the boundary conditions have the form

$$u_k(x_3,0)\equiv 0,\ u_k(\infty,t)\equiv 0,\ k=1,2$$

$$u_1(0,t)=\begin{cases}0, & t\le 0\\ \mu_1=\text{const}, & 0<t\le t_1;\\ 0, & t_1<t\end{cases}\quad u_2(0,t)=\begin{cases}0, & t\le t_2\\ \mu_2=\text{const}, & t_1<t_2<t\end{cases}$$

Only the functions υ_{k2} appear in the solution. Up to time $t = t_2$, the flow is plane; for $t > t_2$ the flow is three-dimensional, owing to the motion of the wall in the Ox direction and the Ox_2 direction

$$u_1(x_3,t)=\begin{cases}\mu_1[1-\Phi(z)], & 0<t\le t_1\\ \mu_1[\Phi(z(\sqrt{1-t_1/t})^{-1})-\Phi(z)], & t_1<t\end{cases}$$

$$u_2(x_3,t)=\mu_2\left[1-\Phi(z(\sqrt{1-t_2/t})^{-1})\right],\ t_2<t,\ z=x_3(\sqrt{\nu t})^{-1}$$

For flow over a permeable plate, when there is a fluid suction at a constant rate u_{30}, the solution can be represented in the form

$$u_k(x_3,t)=v_{k1}(x_3,t)+v_{k2}(x_3,t)+U_k(t)-U_k(0),\ k=1,2$$

$$v_{k1}(x_3,t)=\exp\left(-\frac{u_{30}^2}{4\nu}t\right)\int_0^\infty g(x_3,\xi,t)\varphi_k(\xi)\exp\left[\frac{u_{30}}{2\nu}(x_3-\xi)\right]d\xi$$

$$v_{k2}(x_3,t)=\frac{\nu}{2\sqrt{\pi}}\int_0^t\frac{x_3}{[\nu(t-\tau)]^{3/2}}\exp\left(-\frac{[x_3-u_{30}(t-\tau)]^2}{4\nu(t-\tau)}\right)\times$$
$$\times[\mu_k(\tau)+U_k(0)-U_k(\tau)]d\tau$$
$$u_3=u_{30}$$

We will consider another problem of the fluid flow over a permeable plate which moves in its own plane.

Suppose there is a fluid suction at a constant rate u_{30}. For u_1 we have Eq. (1.5) ($u_{20} = 0$). The boundary conditions are

$$u_1(x_2,x_3,0)=\zeta(x_2,x_3),\ -\infty<x_2<\infty,\ 0\le x_3<\infty$$
$$u_1(x_2,0,t)=\mu(t),\ 0\le t;\ u_1(x_2,\infty,t)=U(t),\ 0\le t$$

Using the variables

$$v=u_1+\int_0^t f(t)dt,\ w=v\exp(-\alpha x_3-\beta t)$$

where $f(t) = \rho^{-1}\partial p/\partial x_1$, $\alpha = u_{30}/(2\nu)$, $\beta = -u_{30}^2/(4\nu)$ and continuing $\zeta(x_2, x_3)$ as an odd function of the variable x_3 in the region $-\infty < x_3 < 0$, we obtain the solution

$$u_1(x_2,x_3,t)=v_1(x_2,x_3,t)+v_2(x_3,t)+U(t)-U(0)$$

$$v_1(x_2,x_3,t)=\exp\left(-\frac{u_{30}^2}{4\nu}t\right)\int_{-\infty}^\infty\int_0^\infty G(x_2,\xi_2,t)g(x_3,\xi_3,t)\times$$

$$\times\zeta(\xi_2,\xi_3)\exp\left(\frac{u_{30}}{2\nu}(x_3-\xi_3)\right)d\xi_3 d\xi_2$$

$$v_2(x_3,t)=\frac{\nu}{2\sqrt{\pi}}\int_0^t \frac{x_3}{[\nu(t-\tau)]^{3/2}}\exp\left(-\frac{[x_3-u_{30}(t-\tau)]^2}{4\nu(t-\tau)}\right)\times$$

$$\times[\mu(\tau)+U(0)-U(\tau)]d\tau$$

$$u_2=0,\ u_3=u_{30}\ (-dU/dt=f(t))$$

3. FLOW IN AN INFINITE TWO-SIDED CORNER

It is required to find a solution of Eq. (1.5), where $u_{20} = 0$, $u_{30} = 0$ in the region $0 < x_2, x_3 < \infty$, $0 < t$, which satisfies the conditions

$$u_1(x_2,x_3,0)=\zeta(x_2,x_3),\ 0\leqslant x_2,x_3<\infty$$

$$u_1(0,x_3,t)=0,\ u_1(x_2,0,t)=0,\ 0\leqslant x_2,x_3<\infty,\ 0\leqslant t$$

$$u_1(\infty,\infty,t)=U(t),\ 0\leqslant t$$

The solution can be written in the form

$$u_1(x_2,x_3,t)=\int_0^\infty\int_0^\infty g(x_2,\xi_2,t)g(x_3,\xi_3,t)\zeta(\xi_2,\xi_3)d\xi_2 d\xi_3+$$

$$+\int_0^t\int_0^\infty\int_0^\infty g(x_2,\xi_2,t-\tau)g(x_3,\xi_3,t-\tau)\frac{dU}{d\tau}d\xi_2 d\xi_3 d\tau$$

$$(dU/dt=-\rho^{-1}\partial p/\partial x_1)$$

4. FLOW BETWEEN PARALLEL PLATES

We will consider a more general problem than that of the flow caused by the sudden motion of a solid boundary relative to a different fixed boundary [3]: it is required to find the solution of Eqs (1.1) in the region $0 < x_3 < l$, $0 < t$, satisfying the conditions

$$u_k(x_3,0)=\zeta_k(x_3),\ 0\leqslant x_3\leqslant l;\ u_k(0,t)=\mu_{1k}(t),\ u_k(l,t)=\mu_{2k}(t),\ 0\leqslant t;\ k=1,2$$

Since $\rho^{-1}\partial p/\partial x_k = f_k(t)$, we have the solution [1]

$$u_k(x_3,t)=\int_0^l G(x_3\xi,t)\chi_k(\xi)d\xi+\int_0^t\int_0^l G(x_3,\xi,t-\tau)h_k(\xi,\tau)d\xi d\tau+$$

$$+\mu_{1k}(t)+\frac{x_3}{l}[\mu_{2k}(t)-\mu_{1k}(t)] \tag{4.1}$$

where

$$\chi_k(\xi)=\zeta(\xi)-\mu_{1k}(0)-\frac{\xi}{l}[\mu_{2k}(0)-\mu_{1k}(0)]$$

$$h_k(\xi,\tau)=f_k(\tau)-\left\{\frac{d\mu_{1k}(\tau)}{d\tau}+\frac{\xi}{l}\left[\frac{d\mu_{2k}(\tau)}{d\tau}-\frac{d\mu_{1k}(\tau)}{d\tau}\right]\right\}$$

$$G(x_3,\xi,t)=\frac{2}{l}\sum_{n=1}^\infty \exp\left[-\left(\frac{n\pi}{l}\right)^2\nu t\right]\sin\frac{n\pi x_3}{l}\sin\frac{n\pi\xi}{l},\ k=1,2$$

The functions ζ_k, μ_{1k}, μ_{2k}, f_k and the quantity l are known. The initial functions $\zeta_k(x_3)$ can be piecewise-continuous and might not be matched to the boundary conditions [1] ($\zeta_k(0) \neq \mu_{1k}(0)$, $\zeta_k(l) \neq \mu_{2k}(0)$, $k = 1, 2$, so that the relations apply in the case where the walls suddenly start to move). The vectors of the velocity of motion of the solid boundaries $\{\mu_{11}, \mu_{12}\}$, $\{\mu_{21}, \mu_{22}\}$ might, generally speaking, be non-collinear.

Suppose that the fluid is initially at rest and moves as a result of the lower boundary instantaneously acquiring a constant velocity U in its plane in the Ox_1 direction, and the upper boundary instantaneously acquiring a constant velocity V in its plane in the Ox_2 direction. From (4.1) in this case we obtain

$$u_1(x_3,t)=U\left(1-\frac{x_3}{l}\right)-\frac{2U}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\exp\left[-\left(\frac{n\pi}{l}\right)^2\nu t\right]\sin\frac{n\pi x_3}{l}$$

$$u_2(x_3,t)=V\frac{x_3}{l}+\frac{2V}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\exp\left[-\left(\frac{n\pi}{l}\right)^2\nu t\right]\sin\frac{n\pi x_3}{l}$$

REFERENCES

1. TIKHONOV, A. N. and SAMARSKII, A. A., *The Equations of Mathematical Physics*. Nauka, Moscow, 1966.
2. KOCHIN, N. Ye., KIBEL', I. A. and ROZE, N. V., *Theoretical Hydromechanics*, Pt. 2. Fizmatgiz, Moscow, 1963.
3. BATCHELOR, G. K., *An Introduction to Fluid Dynamics*. Cambridge University Press, Cambridge, 1967.
4. SCHLICHTING, H., *Boundary Layer Theory*. McGraw-Hill, New York, 1968.

Translated by R.L.